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Identities involving angular momentum operators operating in the space of many-nucleon intrinsic wavefunctions

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Abstract. An angular momentum operator identity which was previously established using matrix representations is re-derived using Feynman's operator differential equation technique.

1. Introduction

The angular momentum operators play an important role in the study of many-body systems. For example, one needs to project good angular momentum states from a given intrinsic state in the study of the collective aspects of a many-nucleon system (Wong 1975, Ullah 1971). The well-known BCH (Baker–Campbell–Hausdorff) (Baker 1905, Campbell 1897, Hausdorff 1906) formulae are not very helpful in recasting the non-commuting angular momentum operators occurring in the projection operator (Löwdin 1964) into a form which is suitable for use with the many-nucleon intrinsic wavefunction. We have shown (Ullah 1971) that matrix representations can be used to recast the product of angular momentum operators, involving a chain of the step-up and step-down operators, into the form of rotation operators.

The purpose of the present work is to use an alternative way of employing differential equation technique proposed by Feynman (1951) to re-derive this important result. The operator differential equation technique has the advantage that it provides a direct approach without recourse to matrix representations for combining the non-commuting quantum mechanical operators.

In § 2 we discuss the essential steps of this formulation. The concluding remarks are presented in § 3.

2. Formulation

In this section, we shall make use of the differential equation technique (Feynman 1951) to recast the product of non-commuting angular momentum operators in a form which is suitable for use with the many-nucleon intrinsic wavefunction.

Let us consider the so-called ladder operator (Wong 1975, Ullah 1971)

$$f(\lambda) = (\exp \lambda J_-)(\exp -\lambda J_+) \tag{1}$$

where J_-, J_+ are the usual step-down and step-up operators respectively and λ is a parameter. The ladder operator enables a simple calculation of the matrix elements of the angular momentum projection operator as given by Löwdin (1964). Since it is the rotation operator $(\exp -i\alpha J_z)(\exp -i\beta J_y)(\exp -i\gamma J_z)$ which can be evaluated easily in the basis of many-nucleon intrinsic wavefunctions, we would like to recast the ladder operator given by expression (1) in this form and express the Euler parameters α, β, γ in terms of λ.

Let us establish the differential equation satisfied by the operator $f(\lambda)$. Differentiating equation (1) with respect to λ, we can write

$$\mathrm{d}f/\mathrm{d}\lambda = [J_- - (\exp \lambda J_-)J_+(\exp -\lambda J_-)]f.$$

Using the operator identity

$$(\exp -S)O(\exp S) = O + [0, S] + \tfrac{1}{2!}[[0, S], S] + \ldots$$

and the commutation relations for the operators J_-, J_+, we can express $\mathrm{d}f/\mathrm{d}\lambda$ as

$$\mathrm{d}f/\mathrm{d}\lambda = [\lambda^2 J_x - i(2+\lambda^2)J_y + 2\lambda J_z]f \tag{2a}$$

with the boundary condition $f(0) = 1$.

It is obvious that the differential equation technique will remain simple so long as the commutators of the operators under consideration can be handled in a simple way as is the case with the angular momentum operators of expression (2*a*).

A very general solution of differential equations like equation (2*a*) has been given by Wei and Norman (1963) in the form of a product of three exponential operators as

$$[\exp(g_1(\lambda)J_x)][\exp(g_2(\lambda)J_y)][\exp(g_3(\lambda)J_z)].$$

Alternatively, this solution can also be obtained by using the integral representation of the derivative of the exponential operator as shown by Wilcox (1967). Since our purpose is to recast the operator f in the form of a rotation operator, we do not use this solution but proceed instead in the following way.

Since $f(\lambda)$ does not commute with the angular momentum operators, we first write one more differential equation for f with $f(\lambda)$ appearing on the left-hand side of the square bracket. It is given by

$$\mathrm{d}f/\mathrm{d}\lambda = f[-\lambda^2 J_x - i(2+\lambda^2)J_y + 2\lambda J_z]. \tag{2b}$$

Now since the angular momentum operators obey Lie algebra, the operator $f(\lambda)$ can be recast in the desired form of the rotation operator as

$$f(\lambda) = (\exp -i\alpha J_z)(\exp -i\beta J_y)(\exp -i\gamma J_z) \tag{3}$$

where α, β, γ are functions of λ. To determine these parameters, we differentiate expression (3) with respect to λ and use the commutation relations for the angular momentum operators to give

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \left[i\left(\sin\alpha\frac{\mathrm{d}\beta}{\mathrm{d}\lambda} - \cos\alpha\sin\beta\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right)J_x - i\left(\cos\alpha\frac{\mathrm{d}\beta}{\mathrm{d}\lambda} + \sin\alpha\sin\beta\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right)J_y - i\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\lambda} + \cos\beta\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right)J_z\right]f. \tag{4}$$

Comparing expressions (2a) and (4) we get the following set of differential equations for the Euler parameters α, β, γ:

$$\sin\alpha\frac{d\beta}{d\lambda}-\cos\alpha\sin\beta\frac{d\gamma}{d\lambda}=-i\lambda^2 \tag{5a}$$

$$\cos\alpha\frac{d\beta}{d\lambda}+\sin\alpha\sin\beta\frac{d\gamma}{d\lambda}=2+\lambda^2 \tag{5b}$$

$$\frac{d\alpha}{d\lambda}+\cos\beta\frac{d\gamma}{d\lambda}=2i\lambda. \tag{5c}$$

As mentioned earlier the operator f in expressions (2) does not commute with the operators in the square bracket. Because of this non-commutativity, one can get one more set of equations for α, β, γ if one writes the operator f on the left-hand-side of the square bracket as was done earlier. These equations are given by

$$\sin\gamma\frac{d\beta}{d\lambda}-\cos\gamma\sin\beta\frac{d\alpha}{d\lambda}=-i\lambda^2 \tag{6a}$$

$$\cos\gamma\frac{d\beta}{d\lambda}+\sin\gamma\sin\beta\frac{d\alpha}{d\lambda}=2+\lambda^2 \tag{6b}$$

$$\frac{d\gamma}{d\lambda}+\cos\beta\frac{d\alpha}{d\lambda}=2i\lambda. \tag{6c}$$

Equations (5) and (6) can be rewritten as

$$\frac{d\gamma}{d\lambda}=\frac{d\alpha}{d\lambda}, \qquad \frac{d\alpha}{d\lambda}=i\lambda\sec^2\tfrac{1}{2}\beta, \qquad \frac{d}{d\lambda}(\sin\tfrac{1}{2}\beta)=(1+\lambda^2-\sin^2\tfrac{1}{2}\beta)^{1/2}. \tag{7a, b, c}$$

Using the boundary condition $f(0)=1$, we get the following solution for the Euler parameters α, β, γ:

$$\gamma=\alpha, \qquad \exp(\tfrac{1}{2}i\alpha)=(1-\lambda^2)^{1/4}, \qquad \cos\tfrac{1}{2}\beta=(1-\lambda^2)^{1/2}. \tag{8a, b, c}$$

We would now like to add that if we have a more general operator of the form $(\exp\rho J_-)(\exp-\omega J_+)$ we can again recast it in the form of the rotation operator (3). To calculate the Euler parameters α, β, γ as a function of ρ, ω we shall have to replace ρ, ω by $\lambda\rho$, $\lambda\omega$ and use the same differential equation technique which we have outlined above. The solution can be written as

$$\begin{aligned}&\exp(\tfrac{1}{2}i\alpha)=[(\rho/\omega)(1-\rho\omega)]^{1/4},\\&\exp(\tfrac{1}{2}i\gamma)=[(\omega/\rho)(1-\rho\omega)]^{1/4},\\&\cos\tfrac{1}{2}\beta=(1-\rho\omega)^{1/2}.\end{aligned} \tag{9a, b, c}$$

3. Concluding remarks

We have shown that the differential equation technique can be used to show that the matrix element identity involving angular momentum operators (Ullah 1971) is also an operator identity. This technique is quite general and can also be applied to recast the product of non-commuting operators into some other convenient form, which is allowed by the commutator algebra of these operators. We have shown that non-commutativity of the operator f with the operator in the square brackets in equation (2a) can be used to obtain additional differential equations for the Euler parameters α, β, γ. Since the differential equations for Euler parameters are non-linear, the additional differential equations are very helpful in arriving at their solutions and also provide a check on the final solutions. Earlier work in this field has not emphasized this important point.

In § 2 we considered recasting exponential functions involving linear functions of the angular momentum operators in the exponent. In the case when the exponent is not linear but involves some power of angular momentum operators, then certain tricks may be used to bring this to a linear form and the differential equation technique may then be used to recast it into the desired form (Ullah and Gupta 1972).

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